

Contact metric (κ, μ) -spaces as bi-Legendrian manifolds

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Abstract

We regard a contact metric manifold whose Reeb vector field belongs to the (κ, μ) -nullity distribution as a bi-Legendrian manifold and we study its canonical bi-Legendrian structure. Then we characterize contact metric (κ, μ) -spaces in terms of a canonical connection which can be naturally defined on them.

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1 Introduction

Contact metric (κ, μ) -spaces, introduced in [2] by D. E. Blair, T. Kouforgiorgos and B. J. Papantoniou, are those contact metric manifolds (M, ϕ, ξ, η, g) for which the Reeb vector field ξ belongs to the (κ, μ) -nullity distribution, i.e. satisfies, for all vector fields V and W on M ,

$$R_{VW}\xi = \kappa(\eta(W)V - \eta(V)W) + \mu(\eta(W)hV - \eta(V)hW), \quad (1.1)$$

for some real numbers κ and μ , where $2h$ is the Lie derivative of ϕ in the direction of ξ . This definition can be regarded as a generalization both of the Sasakian condition $R_{VW}\xi = \eta(W)V - \eta(V)W$ and of those contact metric manifolds verifying $R_{VW}\xi = 0$ which were studied by D. E. Blair in [1].

Recently contact metric (κ, μ) -spaces have been studied by various authors ([4], [5], [6], [11], [14], etc.) and several important properties of these manifolds have been discovered. In fact there are many motivations for studying (κ, μ) -spaces: the first is that, in the non-Sasakian case (that is for $\kappa \neq 1$), the condition (1.1) determines the curvature completely;

moreover, while the values of κ and μ change, the form of (1.1) is invariant under \mathcal{D} -homothetic deformations; finally, there are non-trivial examples of these manifolds, the most important being the unit tangent sphere bundle of a Riemannian manifold of constant sectional curvature with the usual contact metric structure.

A complete classification of contact metric (κ, μ) -spaces has been given in [5] by E. Boeckx, who proved also that any non-Sasakian contact metric (κ, μ) -space is locally homogeneous and strongly locally ϕ -symmetric ([4]).

One of the peculiarities of these manifolds is that they give rise to three mutually orthogonal distributions \mathcal{D}_λ , $\mathcal{D}_{-\lambda}$ and $\mathbb{R}\xi$, corresponding to the eigenspaces of the operator h . In particular \mathcal{D}_λ and $\mathcal{D}_{-\lambda}$ define two transverse Legendrian foliations of M so that these manifolds are endowed with a bi-Legendrian structure.

In the same years the theory of Legendrian foliations has been developed by M. Y. Pang, P. Libermann and N. Jayne (cf. [16], [15], [13]), so it seems to be tempting to use the techniques and the language of Legendrian foliations for the study of contact metric (κ, μ) -spaces and to begin the investigation of the interactions between these two areas of the contact geometry. This is what we set out to do in this article.

The paper is organized as follows. After some preliminaries on contact metric manifolds and Legendrian foliations, in § 3 we study the Legendrian foliations canonically defined in any contact metric (κ, μ) -space. We find, for both the foliations, an explicit formula of the invariant Π introduced by Pang for classifying Legendrian foliations (cf. [16]) and we see that the Legendrian foliations in question are, according to this classification, either non-degenerate or flat. Then we relate these invariants to the invariant I_M used by Boeckx in [5] for classify contact metric (κ, μ) -spaces. In § 4 we attach to any contact metric (κ, μ) -space a linear connection in a canonical way. We study the properties of this connection and, using it, we give an interpretation of the notion of contact metric (κ, μ) -space in terms of bi-Legendrian structures. In particular, we prove the following characterization of contact metric (κ, μ) -spaces.

Theorem 1.1. *A contact metric manifold (M, ϕ, ξ, η, g) is a contact metric (κ, μ) -space if and only if M admits an orthogonal bi-Legendrian structure $(\mathcal{F}, \mathcal{G})$ such that the corresponding bi-Legendrian connection $\bar{\nabla}$ satisfies $\bar{\nabla}\phi = 0$ and $\bar{\nabla}h = 0$. Furthermore, the bi-Legendrian structure $(\mathcal{F}, \mathcal{G})$ coincides with that one determined by the eigenspaces of h .*

This theorem should be compared with the well-known results, obtained by N. Tanaka (cf. [17]) and, independently, S. M. Webster ([22]). They proved that any strongly pseudo-convex CR-manifold admits a unique linear connection $\tilde{\nabla}$ such that the tensors ϕ , η , g are all $\tilde{\nabla}$ -parallel and whose torsion satisfies $\tilde{T}(Z, Z') = 2\Phi(Z, Z')\xi$ for all $Z, Z' \in \Gamma(\mathcal{D})$ and $\tilde{T}(\xi, \phi V) = -\phi\tilde{T}(\xi, V)$ for all $V \in \Gamma(TM)$. In view of this remark and the mentioned theorem of Boeckx that any contact metric (κ, μ) -space is a strongly pseudo-convex CR-manifold, one can see that the connection mentioned in Theorem 1.1 plays the same role for contact metric (κ, μ) -space that the Tanaka-Webster connection has for CR-manifolds. As

we shall see, the connection $\bar{\nabla}$ uniquely determines a contact metric (κ, μ) -space modulo \mathcal{D} -homothetic deformations and it reveals very useful in the study of this kind of contact metric manifolds.

2 Preliminaries

2.1 Contact manifolds

An *almost contact metric manifold* is a $(2n + 1)$ -dimensional Riemannian manifold (M, g) which admits a tensor field ϕ of type $(1, 1)$, a global 1-form η and a global vector field ξ , called *Reeb vector field*, satisfying

$$\eta(\xi) = 1, \quad \phi^2 V = -V + \eta(V)\xi, \quad g(\phi V, \phi W) = g(V, W) - \eta(V)\eta(W), \quad (2.1)$$

for all vector fields V and W on M . Given an almost contact metric manifold one can define a 2-form Φ , called the *fundamental 2-form* of the structure, by $\Phi(V, W) = g(V, \phi W)$. Then we say that (M, ϕ, ξ, η, g) is a *contact metric manifold* if the additional property $d\eta = \Phi$ holds. From (2.1) it can be proven that (cf. [3])

- (i) $\phi\xi = 0, \eta \circ \phi = 0,$
- (ii) $\nabla_\xi \phi = 0$ and $\nabla_\xi \xi = 0,$
- (iii) $\phi|_{\mathcal{D}}$ is an isomorphism,

where ∇ denotes the Levi Civita connection and $\mathcal{D} = \ker(\eta)$ is the $2n$ -dimensional distribution orthogonal to ξ and called the *contact distribution*. It is also easy to prove that for any $X \in \Gamma(\mathcal{D})$ the bracket $[X, \xi]$ still belongs to \mathcal{D} .

In any contact metric manifold the 1-form η satisfies the relation

$$\eta \wedge (d\eta)^n \neq 0 \quad (2.2)$$

everywhere on M . Any $(2n + 1)$ -dimensional smooth manifold which carries a global 1-form satisfying (2.2) is called a *contact manifold*. Thus any contact metric manifold is a contact manifold. Conversely, it is well-known that any contact manifold admits a compatible contact metric structure (ϕ, ξ, η, g) . It should be remarked that (2.2) implies that the contact distribution \mathcal{D} is never integrable.

Given a contact metric manifold, we can define a tensor field h by $h = \frac{1}{2}\mathcal{L}_\xi \phi$, \mathcal{L} denoting the Lie differentiation. It can be shown (cf. [3]) that h is a trace-free, symmetric operator verifying $h\xi = 0, \phi h = -h\phi$ and

$$\nabla_V \xi = -\phi h V - \phi V \quad (2.3)$$

for all $V \in \Gamma(TM)$. Moreover ξ is Killing if and only if h vanishes identically; in this case we say that (M, ϕ, ξ, η, g) is a *K-contact manifold*.

On a contact metric manifold M one can define an almost complex structure J on the product manifold $M \times \mathbb{R}$ by setting $J(V, f \frac{d}{dt}) = (\phi V - f\xi, \eta(V) \frac{d}{dt})$, where V is a vector field tangent to M and f a function on $M \times \mathbb{R}$. If the almost complex structure J is integrable then (M, ϕ, ξ, η, g) is said to be *Sasakian*. It is well-known that each of the following conditions characterizes Sasakian manifolds

$$(\nabla_V \phi)W = g(V, W)\xi - \eta(W)V \quad (2.4)$$

$$R_{VW}\xi = \eta(W)V - \eta(V)W \quad (2.5)$$

for all vector fields V and W on M . A generalization of the condition (2.5) leads to the notion of (κ, μ) -manifold. If the curvature tensor field of a contact metric manifold satisfies (1.1) for some real numbers κ and μ we say that ξ belongs to the (κ, μ) -nullity distribution or, simply, that (M, ϕ, ξ, η, g) is a contact metric (κ, μ) -space. These manifolds were introduced and deeply studied in [2]. Among other things, the authors proved the following results.

Theorem 2.1 ([2]). *Let (M, ϕ, ξ, η, g) be a contact metric manifold with ξ belonging to the (κ, μ) -nullity distribution. Then $\kappa \leq 1$. Moreover, if $\kappa = 1$ then $h = 0$ and (M, ϕ, ξ, η, g) is a Sasakian manifold; if $\kappa < 1$, the contact metric structure is not Sasakian and M admits three mutually orthogonal integrable distributions $\mathcal{D}_0 = \mathbb{R}\xi$, \mathcal{D}_λ and $\mathcal{D}_{-\lambda}$ corresponding to the eigenspaces of h , where $\lambda = \sqrt{1 - \kappa}$.*

Theorem 2.2 ([2]). *Let (M, ϕ, ξ, η, g) be a contact metric manifold with ξ belonging to the (κ, μ) -nullity distribution. Then the following relation hold, for any $X, Y \in \Gamma(TM)$,*

$$\begin{aligned} (\nabla_X \phi)Y &= g(X, Y + hY)\xi - \eta(Y)(X + hX), \\ (\nabla_X h)Y &= ((1 - \kappa)g(X, \phi Y) + g(X, \phi hY))\xi + \eta(Y)(h(\phi X + \phi hX)) - \mu \phi hY. \end{aligned}$$

Blair, Koufogiorgos and Papantoniou proven also that the (κ, μ) -nullity condition remains unchanged under \mathcal{D} -homothetic deformations. The concept of \mathcal{D} -homothetic deformation for a contact metric manifold (M, ϕ, ξ, η, g) has been introduced by S. Tanno in [18] and then intensively studied by many authors. We recall that, given a real positive number a , by a \mathcal{D} -homothetic deformation of constant a we mean a change of the structure tensors in the following way:

$$\tilde{\phi} = \phi, \quad \tilde{\eta} = a\eta, \quad \tilde{\xi} = \frac{1}{a}\xi, \quad \tilde{g} = ag + a(a - 1)\eta \otimes \eta. \quad (2.6)$$

In [2] the authors proven that if M is a contact metric manifold whose Reeb vector field belongs to the (κ, μ) -nullity distribution then for the contact metric manifold $(M, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$

the same property holds. Precisely $\tilde{\xi}$ belongs to the $(\tilde{\kappa}, \tilde{\mu})$ -nullity distribution where

$$\tilde{\kappa} = \frac{\kappa + a^2 - 1}{a^2}, \quad \tilde{\mu} = \frac{\mu + 2a - 2}{a}.$$

2.2 Legendrian foliations

A *Legendrian distribution* on a contact manifold (M^{2n+1}, η) is defined by an n -dimensional subbundle L of the contact distribution such that $d\eta(X, X') = 0$ for all $X, X' \in \Gamma(L)$. When L is integrable, it defines a *Legendrian foliation* of (M^{2n+1}, η) . Legendrian foliations have been extensively investigated in recent years from various points of views (cf. [16], [15], [13], [8], etc.). In particular Pang provided a classification of Legendrian foliations by means of a bilinear symmetric form $\Pi_{\mathcal{F}}$ on the tangent bundle of the foliation, defined by $\Pi_{\mathcal{F}}(X, X') = -(\mathcal{L}_X \mathcal{L}_{X'} \eta)(\xi) = -\eta([X', [X, \xi]])$. He called a Legendrian foliation \mathcal{F} *non-degenerate*, *degenerate* or *flat* according to the circumstance that the bilinear form $\Pi_{\mathcal{F}}$ is non-degenerate, degenerate or vanishes identically, respectively. In terms of an associated metric g , $\Pi_{\mathcal{F}}$ is given by

$$\Pi_{\mathcal{F}}(X, X') = 2g([\xi, X], \phi X'). \quad (2.7)$$

The last formula provides a geometrical interpretation of this classification:

Lemma 2.1 ([13]). *Let (M, ϕ, ξ, η, g) be a contact metric manifold and let \mathcal{F} be a foliation on it. Then*

- (i) \mathcal{F} is flat if and only if $[\xi, X] \in \Gamma(T\mathcal{F})$ for all $X \in \Gamma(T\mathcal{F})$,
- (ii) \mathcal{F} is degenerate if and only if there exist $X \in \Gamma(T\mathcal{F})$ such that $[\xi, X] \in \Gamma(T\mathcal{F})$,
- (iii) \mathcal{F} is non-degenerate if and only if $[\xi, X] \notin \Gamma(T\mathcal{F})$ for all $X \in \Gamma(T\mathcal{F})$.

Given a compatible contact metric structure (ϕ, ξ, η, g) and a Legendrian distribution L on M , we may consider the distribution $Q = \phi L$. It can be proven (cf. [13]) that Q is a Legendrian distribution on M which in general is not integrable, even if L is; it is called the *conjugate Legendrian distribution* of L , and the tangent bundle of M splits as the orthogonal sum $TM = L \oplus Q \oplus \mathbb{R}\xi$. When both L and Q are integrable, they defines two orthogonal Legendrian foliations \mathcal{F} and \mathcal{G} on M , and the pair $(\mathcal{F}, \mathcal{G})$ is an example of a *bi-Legendrian structure* on M . More in general a bi-Legendrian structure is a pair of two complementary, not necessarily orthogonal, Legendrian foliations on M .

In [7] it has been attached to any contact manifold (M^{2n+1}, η) endowed with a pair of two complementary Legendrian distributions (L, Q) a linear connection $\bar{\nabla}$ uniquely

determined by the following properties:

$$\begin{aligned}
& \text{(i)} \quad \bar{\nabla} L \subset L, \quad \bar{\nabla} Q \subset Q, \quad \bar{\nabla}(\mathbb{R}\xi) \subset \mathbb{R}\xi, \\
& \text{(ii)} \quad \bar{\nabla} d\eta = 0, \\
& \text{(iii)} \quad \bar{T}(X, Y) = 2d\eta(X, Y)\xi, \text{ for all } X \in \Gamma(L), Y \in \Gamma(Q), \\
& \quad \bar{T}(V, \xi) = [\xi, V_L]_Q + [\xi, V_Q]_L, \text{ for all } V \in \Gamma(TM),
\end{aligned} \tag{2.8}$$

where \bar{T} denotes the torsion tensor of $\bar{\nabla}$ and V_L and V_Q the projections of V onto the subbundles L and Q of TM , respectively. Such a connection is called the *bi-Legendrian connection* associated to the pair (L, Q) and it is defined as follows (cf. [7]). For all $V \in \Gamma(TM)$, $X \in \Gamma(L)$ and $Y \in \Gamma(Q)$, $\bar{\nabla}_V X := H(V_L, X)_L + [V_Q, X]_L + [V_{\mathbb{R}\xi}, X]_L$, $\bar{\nabla}_V Y := H(V_Q, Y)_Q + [V_L, Y]_Q + [V_{\mathbb{R}\xi}, Y]_Q$ and $\bar{\nabla}\xi = 0$, where H denotes the operator such that, for all $V, W \in \Gamma(TM)$, $H(V, W)$ is the unique section of \mathcal{D} satisfying $i_{H(V, W)} d\eta|_{\mathcal{D}} = (\mathcal{L}_V i_W d\eta)|_{\mathcal{D}}$. Further properties of this connection are collected in the following proposition.

Proposition 2.1 ([7]). *Let (M, η) be a contact manifold endowed with two complementary Legendrian distributions L and Q and let $\bar{\nabla}$ denote the corresponding bi-Legendrian connection. Then the 1-form η and the vector field ξ are $\bar{\nabla}$ -parallel and the complete expression of the torsion tensor field is given by $\bar{T}(X, X') = -[X, X']_Q$ for all $X, X' \in \Gamma(L)$ and $\bar{T}(Y, Y') = -[Y, Y']_L$ for all $Y, Y' \in \Gamma(Q)$.*

Now consider a contact metric manifold (M, ϕ, ξ, η, g) endowed with two complementary Legendrian distributions L and Q . The definition of the corresponding bi-Legendrian connection does not involve the compatible metric g , however it makes sense to find conditions which ensure $\bar{\nabla}$ being a metric connection at least when Q is orthogonal to L . This problem has been solved in [9] where the author proves the following result.

Proposition 2.2. *Let (M, ϕ, ξ, η, g) be a contact metric manifold and L be a Legendrian distribution on M . Let $Q = \phi L$ be the conjugate Legendrian distribution of L and $\bar{\nabla}$ the associated bi-Legendrian connection. Then the following statements are equivalent:*

- (i) $\bar{\nabla} g = 0$;
- (ii) $\bar{\nabla} \phi = 0$;
- (iii) g is a bundle-like metric with respect both to the distribution $L \oplus \mathbb{R}\xi$ and to $Q \oplus \mathbb{R}\xi$;
- (iv) $\bar{\nabla}_X X' = (\phi[X, \phi X'])_L$ for all $X, X' \in \Gamma(L)$, $\bar{\nabla}_Y Y' = (\phi[Y, \phi Y'])_Q$ for all $Y, Y' \in \Gamma(Q)$ and the operator h maps the subbundle L onto L and the subbundle Q onto Q .

Furthermore, assuming L and Q integrable, (i)–(iv) are equivalent to the total geodesicity of the Legendrian foliations defined by L and Q

By a *bi-Legendrian manifold* we mean a contact manifold endowed with two transversal Legendrian foliations. In particular, in this paper we deal with contact metric manifolds foliated by two mutually orthogonal Legendrian foliations. With regard to this, it will be useful in the sequel to prove the following lemma, which states essentially that in a bi-Legendrian manifold the operator h is deeply linked to the given bi-Legendrian structure. This is just the starting point of our work.

Lemma 2.2. *Let \mathcal{F} and \mathcal{G} two mutually orthogonal Legendrian foliations on the contact metric manifold (M, ϕ, ξ, η, g) . Then for all $X, X' \in \Gamma(T\mathcal{F})$*

$$\Pi_{\mathcal{F}}(X, X') - \Pi_{\mathcal{G}}(\phi X, \phi X') = 4g(hX, X'). \quad (2.9)$$

Proof. Since, by the orthogonality between \mathcal{F} and \mathcal{G} we have $\phi(T\mathcal{F}) = T\mathcal{G}$, using (2.7) we have

$$\begin{aligned} \Pi_{\mathcal{F}}(X, X') - \Pi_{\mathcal{G}}(\phi X, \phi X') &= 2g([\xi, X], \phi X') - 2g([\xi, \phi X], \phi^2 X') \\ &= 2g([\xi, \phi X], X') - 2g(\phi[\xi, X], X') \\ &= 4g(hX, X'). \end{aligned}$$

□

Corollary 2.1. *If M is K -contact then \mathcal{F} and \mathcal{G} belong to the same class according to the above Pang's classification.*

Corollary 2.2. *If \mathcal{F} and \mathcal{G} are both flat then M is K -contact.*

3 On the bi-Legendrian structure associated to a contact metric (κ, μ) -space

Let (M, ϕ, ξ, η, g) be a contact metric manifold such that ξ belongs to the (κ, μ) -nullity distribution. By Theorem 2.1 the orthogonal distributions \mathcal{D}_{λ} and $\mathcal{D}_{-\lambda}$ defined by the eigenspaces of h are involutive and define on M two orthogonal Legendrian foliations which we denote by \mathcal{F}_{λ} and $\mathcal{F}_{-\lambda}$, respectively. In this section we begin the study of the bi-Legendrian manifold $(M, \mathcal{F}_{\lambda}, \mathcal{F}_{-\lambda})$.

Proposition 3.1. *Let (M, ϕ, ξ, η, g) be a contact metric (κ, μ) -space which is not K -contact. Then the Legendrian foliations \mathcal{F}_{λ} and $\mathcal{F}_{-\lambda}$ are either non-degenerate or flat. More precisely, \mathcal{F}_{λ} (respectively, $\mathcal{F}_{-\lambda}$) is flat if and only if $\kappa + \mu\lambda - (\lambda + 1)^2 = 0$ (respectively, $\kappa - \mu\lambda - (\lambda - 1)^2 = 0$), otherwise being non-degenerate.*

Proof. Let $X \in \Gamma(\mathcal{D}_\lambda)$. Then by (1.1) we have

$$R_{X\xi}\xi = \kappa X + \mu hX = (\kappa + \mu\lambda) X.$$

On the other hand, using (2.3),

$$\begin{aligned} R_{X\xi}\xi &= -\nabla_\xi \nabla_X \xi - \nabla_{[X,\xi]}\xi \\ &= \nabla_\xi \phi X + \lambda \nabla_\xi \phi X + \phi[X,\xi] + \phi h[X,\xi] \\ &= X - \lambda X - [\phi X, \xi] + \lambda X - \lambda^2 X - \lambda[\phi X, \xi] + \phi[X,\xi] + \phi h[X,\xi] \\ &= (\lambda + 1)^2 X - \lambda \phi[X,\xi] + \phi h[X,\xi], \end{aligned}$$

so that

$$\phi h[X,\xi] = \lambda \phi[X,\xi] + (\kappa + \mu\lambda - (\lambda + 1)^2)X$$

hence, applying ϕ and taking into account that $[X,\xi] \in \Gamma(\mathcal{D})$,

$$-h[X,\xi] = -\lambda[X,\xi] + (\kappa + \mu\lambda - (\lambda + 1)^2)\phi X.$$

Decomposing $[X,\xi]$ in the directions of \mathcal{D}_λ and $\mathcal{D}_{-\lambda}$ we obtain

$$-h([X,\xi]_{\mathcal{D}_\lambda} + [X,\xi]_{\mathcal{D}_{-\lambda}}) = -\lambda[X,\xi] + (\kappa + \mu\lambda - (\lambda + 1)^2)\phi X,$$

from which it follows that

$$2\lambda[X,\xi]_{\mathcal{D}_{-\lambda}} = (\kappa + \mu\lambda - (\lambda + 1)^2)\phi X \quad (3.1)$$

and we conclude, according to Lemma 2.1, that \mathcal{F}_λ is either flat or non-degenerate. The first case occurs if and only if $\kappa + \mu\lambda - (\lambda + 1)^2 = 0$ and the second when $\kappa + \mu\lambda - (\lambda + 1)^2 \neq 0$. In a similar way one can prove the analogous results for $\mathcal{F}_{-\lambda}$. \square

Remark 3.1. From Corollary 2.1 it follows that the bi-Legendrian structure $(\mathcal{F}_\lambda, \mathcal{F}_{-\lambda})$ is flat if and only if $\kappa = 1$ and hence M is Sasakian. This can be also prove in a direct way observing that, according to Proposition 3.1, the functions $f(\kappa, \mu) = \kappa + \mu\lambda - \lambda(\lambda + 1)^2 = 2(\kappa - 1) + (\mu - 2)\sqrt{1 - \kappa}$ and $g(\kappa, \mu) = \kappa - \mu\lambda - \lambda(\lambda - 1)^2 = 2(\kappa - 1) + (2 - \mu)\sqrt{1 - \kappa}$ both vanish if and only if $\kappa = 1$.

Proposition 3.1 extends and improves the results obtained in [12] for contact metric manifolds for which ξ belongs to the κ -nullity distribution (cf. [19]), i.e. the Levi Civita connection of g satisfies $R_{VW}\xi = \kappa(\eta(W)V - \eta(V)W)$. In [12] the author proven that the bi-Legendrian structure associated to such contact metric manifolds is non-degenerate; we recall that in his proof he used the fact that the non-degenerate plane sections containing ξ have constant sectional curvature and this last property does not hold for contact metric (κ, μ) -spaces, as it is been proven in [2].

We remark also that from the proof of Proposition 3.1 it follows an explicit expression of the invariants $\Pi_{\mathcal{F}_\lambda}$ and $\Pi_{\mathcal{F}_{-\lambda}}$ of the Legendrian foliations \mathcal{F}_λ and $\mathcal{F}_{-\lambda}$. More precisely, from (3.1) and (2.7) one can prove the following proposition.

Proposition 3.2. *Let (M, ϕ, ξ, η, g) be a contact metric (κ, μ) -space which is not K -contact. Then the canonical invariants associated to the Legendrian foliations \mathcal{F}_λ and $\mathcal{F}_{-\lambda}$ are given by*

$$\Pi_{\mathcal{F}_\lambda} = \frac{(\lambda + 1)^2 - \kappa - \mu\lambda}{\lambda} g|_{\mathcal{F}_\lambda \times \mathcal{F}_\lambda} \quad \text{and} \quad \Pi_{\mathcal{F}_{-\lambda}} = \frac{-(\lambda - 1)^2 + \kappa - \mu\lambda}{\lambda} g|_{\mathcal{F}_{-\lambda} \times \mathcal{F}_{-\lambda}}, \quad (3.2)$$

respectively.

It should be remarked that the pair $(\Pi_{\mathcal{F}_\lambda}, \Pi_{\mathcal{F}_{-\lambda}})$ is an invariant of the contact metric (κ, μ) -space in question up to \mathcal{D} -homothetic deformations. Indeed let $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ be a \mathcal{D} -homothetic deformation of (ϕ, ξ, η, g) . Then first of all since $\tilde{h} = \frac{1}{2}\mathcal{L}_{\tilde{\xi}}\tilde{\phi} = \frac{1}{a}h$ (cf. [2]), the eigenvalues of \tilde{h} are $\pm\tilde{\lambda} = \pm\frac{1}{a}\lambda$, apart from 0. It follows that the eigenspaces $\mathcal{D}_{\tilde{\lambda}}$ and $\mathcal{D}_{-\tilde{\lambda}}$ coincide with \mathcal{D}_λ and $\mathcal{D}_{-\lambda}$ respectively. Next, for all $X, X' \in \Gamma(\mathcal{D}_{\tilde{\lambda}}) = \Gamma(\mathcal{D}_\lambda)$ we have $\Pi_{\mathcal{F}_{\tilde{\lambda}}}(X, X') = -\tilde{\eta}([X', [X, \tilde{\xi}]]) = -a\eta(\frac{1}{a}[X', [X, \xi]]) = \Pi_{\mathcal{F}_\lambda}(X, X')$. Analogously one can prove that $\Pi_{\mathcal{F}_{-\tilde{\lambda}}} = \Pi_{\mathcal{F}_{-\lambda}}$. Moreover, it should be observed that the invariant $\Pi_{\mathcal{F}}$ of any Legendrian foliation \mathcal{F} depend only on the Legendrian foliation and on the contact form η and not on the associated metric g . In particular the function

$$\frac{\Pi_{\mathcal{F}_\lambda}(X, X') + \Pi_{\mathcal{F}_{-\lambda}}(\phi X, \phi X')}{\Pi_{\mathcal{F}_\lambda}(X, X') - \Pi_{\mathcal{F}_{-\lambda}}(\phi X, \phi X')}, \quad (3.3)$$

for all $X, X' \in \Gamma(\mathcal{D}_\lambda)$ such that $\Pi_{\mathcal{F}_\lambda}(X, X') \neq 0$ (or, equivalently, $g(X, X') \neq 0$), is an invariant of the bi-Legendrian manifold M up to \mathcal{D} -homothetic deformations and it does not depend on the vector fields $X, X' \in \Gamma(\mathcal{D}_\lambda)$. Indeed, after a straightforward computation, taking into account Lemma 2.2, (3.2) and (2.1), one can find that (3.3) is a constant and, more precisely, it is given by

$$\frac{\Pi_{\mathcal{F}_\lambda}(X, X') + \Pi_{\mathcal{F}_{-\lambda}}(\phi X, \phi X')}{\Pi_{\mathcal{F}_\lambda}(X, X') - \Pi_{\mathcal{F}_{-\lambda}}(\phi X, \phi X')} = \frac{1 - \frac{\mu}{2}}{4\sqrt{1 - \kappa}} = \frac{1}{4}I_M,$$

where I_M is the invariant introduced by Boeckx in [5] for classifying contact metric (κ, μ) -spaces. In particular if \mathcal{F}_λ (respectively $\mathcal{F}_{-\lambda}$) is flat then I_M attains the value 4 (respectively -4). Moreover, we can also give an explicit formula for the constant μ in terms of Legendrian foliations

$$\mu = \frac{\Pi_{\mathcal{F}_\lambda}(X, X')}{g(hX, X')} = \frac{\Pi_{\mathcal{F}_\lambda}(X, X')}{\lambda g(X, X')} \quad (3.4)$$

for all $X, X' \in \Gamma(\mathcal{D}_\lambda)$ such that $g(X, X') \neq 0$.

4 An interpretation of contact metric (κ, μ) -spaces

Let (M, ϕ, ξ, η, g) be a contact metric (κ, μ) -space. We can attach to the bi-Legendrian structure $(\mathcal{F}_\lambda, \mathcal{F}_{-\lambda})$ the corresponding bi-Legendrian connection $\bar{\nabla}$, that is the unique linear connection on M such that (2.8) hold. Furthermore we have the following result.

Proposition 4.1. *Let (M, ϕ, ξ, η, g) be a contact metric (κ, μ) -space and let $\bar{\nabla}$ be the bi-Legendrian connection associated to M . Then the tensors ϕ , h and g are $\bar{\nabla}$ -parallel. Moreover, for the torsion tensor of $\bar{\nabla}$ we have $\bar{T}(Z, Z') = 2\Phi(Z, Z')\xi$ for all $Z, Z' \in \Gamma(\mathcal{D})$.*

Proof. A well-known property about \mathcal{F}_λ and $\mathcal{F}_{-\lambda}$ is that they are totally geodesic foliations (cf. [2]). Thus applying Proposition 2.2 we get $\bar{\nabla}g = 0$ and $\bar{\nabla}\phi = 0$. Next, for all $V \in \Gamma(TM)$, $X \in \Gamma(\mathcal{D}_+)$, $Y \in \Gamma(\mathcal{D}_-)$, we have

$$\begin{aligned} (\bar{\nabla}_V h)X &= \bar{\nabla}_V hX - h\bar{\nabla}_V X = \bar{\nabla}_V(\lambda X) - \lambda\bar{\nabla}_V X = 0, \\ (\bar{\nabla}_V h)Y &= \bar{\nabla}_V hY - h\bar{\nabla}_V Y = \bar{\nabla}_V(-\lambda Y) + \lambda\bar{\nabla}_V Y = 0, \end{aligned}$$

because $\bar{\nabla}$ preserves \mathcal{F}_λ and $\mathcal{F}_{-\lambda}$. Finally, for any $f \in C^\infty(M)$,

$$(\bar{\nabla}_V h)f\xi = \bar{\nabla}_V(h(f\xi)) - h(\bar{\nabla}_V(f\xi)) = -h(f\bar{\nabla}_V\xi) - V(f)h\xi = 0$$

because $\bar{\nabla}\xi = 0$ and $h\xi = 0$. It remains to prove the property about the torsion, but it follows easily from Proposition 2.1 and from the integrability of \mathcal{D}_λ and $\mathcal{D}_{-\lambda}$. \square

Corollary 4.1. *With the assumptions and the notation of Proposition 4.1, the connection $\bar{\nabla}$ is related to the Levi Civita connection of (M, ϕ, ξ, η, g) by the following formula, for all $X, Y \in \Gamma(\mathcal{D})$,*

$$\bar{\nabla}_X Y = \nabla_X Y - \eta(\nabla_X Y)\xi. \quad (4.1)$$

Proof. Since $\bar{\nabla}$ is torsion free along the leaves of the foliations \mathcal{F}_λ and $\mathcal{F}_{-\lambda}$ and, by Proposition 4.1, it is metric, it coincides with the Levi Civita connection along the leaves of \mathcal{F}_λ and $\mathcal{F}_{-\lambda}$. Hence (4.1) holds for all $X, Y \in \Gamma(\mathcal{D}_\lambda)$ or $X, Y \in \Gamma(\mathcal{D}_{-\lambda})$ because \mathcal{F}_λ and $\mathcal{F}_{-\lambda}$ are totally geodesic foliations. Now let $X \in \Gamma(\mathcal{D}_\lambda)$ and $Y \in \Gamma(\mathcal{D}_{-\lambda})$. It is well-known (cf. [2]) that $\nabla_X Y \in \Gamma(\mathcal{D}_{-\lambda} \oplus \mathbb{R}\xi)$. For all $Y' \in \Gamma(\mathcal{D}_{-\lambda})$, using $\bar{\nabla}g = 0$, we have

$$\begin{aligned} 2g(\nabla_X Y, Y') &= X(g(Y, Y')) + Y(g(X, Y')) - Y'(g(X, Y)) + g([X, Y], Y') \\ &\quad + g([Y', X], Y) - g([Y, Y'], X) \\ &= X(g(Y, Y')) + g([X, Y], Y') + g([Y', X], Y) \\ &= X(g(Y, Y')) - g([X, Y']_{\mathcal{D}_{-\lambda}}, Y) + g([X, Y]_{\mathcal{D}_{-\lambda}}, Y') \\ &= 2g([X, Y]_{\mathcal{D}_{-\lambda}}, Y') \\ &= 2g(\bar{\nabla}_X Y, Y'), \end{aligned}$$

from which it follows that $\bar{\nabla}_X Y = (\nabla_X Y)_{\mathcal{D}_{-\lambda}}$ and hence (4.1). Analogously one can prove (4.1) for $X \in \Gamma(\mathcal{D}_{-\lambda})$ and $Y \in \Gamma(\mathcal{D}_\lambda)$. \square

Now we examine in a certain sense an "inverse" problem. We start with a bi-Legendrian structure on an arbitrary contact metric manifold M and we ask whether M is a contact metric (κ, μ) -space for some $\kappa, \mu \in \mathbb{R}$.

Theorem 4.1. *Let (M, ϕ, ξ, η, g) be a contact metric manifold, non K -contact, endowed with two orthogonal Legendrian foliations \mathcal{F} and \mathcal{G} and suppose that the bi-Legendrian connection corresponding to $(\mathcal{F}, \mathcal{G})$ satisfies $\bar{\nabla}\phi = 0$ and $\bar{\nabla}h = 0$. Then (M, ϕ, ξ, η, g) is a contact metric (κ, μ) -space. Furthermore, the bi-Legendrian structure $(\mathcal{F}, \mathcal{G})$ coincides with that one determined by the eigenspaces of h .*

Proof. Firstly we prove that under our assumptions (4.1) holds. Since, by Proposition 2.2, $\bar{\nabla}g = 0$ and $\bar{T}(X, X') = 0$, $\bar{T}(Y, Y') = 0$ for all $X, X' \in \Gamma(T\mathcal{F})$ and $Y, Y' \in \Gamma(T\mathcal{G})$, it follows immediately that the bi-Legendrian connection and the Levi Civita connection coincide along the leaves of \mathcal{F} and \mathcal{G} . Moreover, for all $X \in \Gamma(T\mathcal{F})$ and $Y \in \Gamma(T\mathcal{G})$ $\nabla_X Y \in \Gamma(T\mathcal{G} \oplus \mathbb{R}\xi)$ because for all $X' \in \Gamma(T\mathcal{F})$

$$g(\nabla_X Y, X') = X(g(Y, X')) - g(Y, \nabla_X X') = 0$$

since \mathcal{F} , as well as \mathcal{G} , is totally geodesic by Proposition 2.2. Then one can argue as in the proof of Corollary 4.1 and prove that

$$\nabla_Z Z' = \bar{\nabla}_Z Z' + \eta(\nabla_Z Z')\xi \quad (4.2)$$

for all $Z, Z' \in \Gamma(\mathcal{D})$. Now for all $X, Y, Z \in \Gamma(\mathcal{D})$ we have, applying (4.2),

$$\begin{aligned} g((\nabla_X h)Y, Z) &= g(\nabla_X hY - h\nabla_X Y, Z) \\ &= g(\bar{\nabla}_X hY + \eta(\nabla_X hY)\xi - h\bar{\nabla}_X Y - \eta(\nabla_X Y)h\xi, Z) \\ &= g((\bar{\nabla}_X h)Y, Z) + \eta(\nabla_X hY)\eta(Z) \\ &= g((\bar{\nabla}_X h)Y, Z) = 0, \end{aligned}$$

since, by assumption, $\bar{\nabla}h = 0$. Thus the tensor field h is η -parallel and so, by [6, Theorem 4], (M, ϕ, ξ, η, g) is a contact metric (κ, μ) -space. For proving the last part of the theorem, suppose by absurd that \mathcal{F} does not coincide with both \mathcal{F}_λ and $\mathcal{F}_{-\lambda}$. Let X be a vector field tangent to \mathcal{F} and decompose it as $X = X_+ + X_-$, with $X_+ \in \Gamma(\mathcal{D}_\lambda)$ and $X_- \in \Gamma(\mathcal{D}_{-\lambda})$. Then we have $hX = h(X_+) + h(X_-) = \lambda X_+ - \lambda X_- = \lambda(X_+ - X_-)$, from which, since by Proposition 2.2 h preserves \mathcal{F} , it follows that $X_+ - X_- \in \Gamma(T\mathcal{F})$. On the other hand also $X_+ + X_- = X \in \Gamma(T\mathcal{F})$, hence X_+ and X_- are both tangent to \mathcal{F} and this is a contradiction. \square

From Theorem 4.1 we get the following characterization of contact metric (κ, μ) -spaces. Here, by an abuse of language, we call Legendrian distribution of an almost contact manifold any n -dimensional subbundle L of the distribution $\mathcal{D} = \ker(\eta)$ such that $d\eta(X, X') = 0$ for all $X, X' \in \Gamma(L)$ and, as in contact metric geometry, $2h$ is defined as the Lie differentiation of the tensor ϕ along the Reeb vector field ξ .

Theorem 4.2. *Let (M, ϕ, ξ, η, g) be an almost contact metric manifold with ξ non-Killing. Then (M, ϕ, ξ, η, g) is a contact metric (κ, μ) -space if and only if it admits two orthogonal conjugate Legendrian distributions L and Q and a linear connection $\tilde{\nabla}$ satisfying the following properties:*

- (i) $\tilde{\nabla}L \subset L, \tilde{\nabla}Q \subset Q,$
- (ii) $\tilde{\nabla}\eta = 0, \tilde{\nabla}d\eta = 0, \tilde{\nabla}g = 0, \tilde{\nabla}h = 0,$
- (iii) $\tilde{T}(Z, Z') = 2\Phi(Z, Z')\xi$ for all $Z, Z' \in \Gamma(\mathcal{D}),$
 $\tilde{T}(V, \xi) = [\xi, V_L]_Q + [\xi, V_Q]_L$ for all $V \in \Gamma(TM),$

where \tilde{T} denotes the torsion tensor field of $\tilde{\nabla}$. Furthermore $\tilde{\nabla}$ is uniquely determined, L and Q are integrable and coincide with the eigenspaces of the operator h .

Proof. The proof is rather obvious in one direction, it is sufficient to take as $\tilde{\nabla}$ the bi-Legendrian connection associated to the bi-Legendrian structure defined by the eigenspaces of h . Now we prove the converse. Note that by (ii) it follows also that ξ is parallel with respect to $\tilde{\nabla}$, because for any $V \in \Gamma(TM)$ $(\tilde{\nabla}_V\eta)\xi = -\eta(\tilde{\nabla}_V\xi) = 0$, so $\tilde{\nabla}_V\xi \in \Gamma(\mathcal{D})$. On the other hand for any $Z \in \Gamma(\mathcal{D})$, since $\tilde{\nabla}$ is a metric connection and preserves the subbundle $\mathcal{D} = L \oplus Q$, we have

$$g(\tilde{\nabla}_V\xi, Z) = V(g(\xi, Z)) - g(\xi, \tilde{\nabla}_VZ) = 0,$$

from which $\tilde{\nabla}_V\xi$ is also orthogonal to \mathcal{D} hence vanishes. Now we can prove the result. We show first that $d\eta = \Phi$, so M is a contact metric manifold. For any $X, X' \in \Gamma(L)$ and $Y, Y' \in \Gamma(Q)$ we have $d\eta(X, X') = 0 = g(X, \phi X')$ and $d\eta(Y, Y') = 0 = g(Y, \phi Y')$. Moreover

$$2\Phi(X, Y)\xi = \tilde{T}(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y]$$

from which

$$2\Phi(X, Y) = g(\tilde{\nabla}_X Y, \xi) - g(\tilde{\nabla}_Y X, \xi) - g([X, Y], \xi). \quad (4.3)$$

Now, $g(\tilde{\nabla}_X Y, \xi) = X(g(Y, \xi)) - g(Y, \tilde{\nabla}_X \xi) = 0$ and, analogously, $g(\tilde{\nabla}_Y X, \xi) = 0$, so that (4.3) becomes

$$2\Phi(X, Y) = -\eta([X, Y]),$$

from which it follows that $d\eta(X, Y) = \Phi(X, Y)$. For concluding that (M, ϕ, ξ, η, g) is a contact metric manifold it remains to check that $d\eta(Z, \xi) = \Phi(Z, \xi)$ for any $Z \in \Gamma(\mathcal{D})$. Indeed $d\eta(Z, \xi) = -\frac{1}{2}\eta([Z, \xi]) = 0 = \Phi(Z, \xi)$ since

$$[Z, \xi] = \tilde{\nabla}_Z \xi - \tilde{\nabla}_\xi Z - \tilde{T}(Z, \xi) = -\tilde{\nabla}_\xi Z - [\xi, Z_L]_Q - [\xi, Z_Q]_L \in \Gamma(\mathcal{D})$$

because of (i). Therefore (M, ϕ, ξ, η, g) is a contact metric manifold endowed with two complementary (in particular orthogonal) Legendrian distributions L and Q , and since $\tilde{\nabla}\xi = 0$ the connection $\tilde{\nabla}$ coincides with the bi-Legendrian connection $\bar{\nabla}$ associated to (L, Q) . This fact and (iii) imply the integrability of L and Q . Indeed for any $X, X' \in \Gamma(L)$ we have

$$[X, X']_Q = -\bar{T}(X, X') = -\tilde{T}(X, X') = -2d\eta(X, X')\xi = 0$$

and

$$g([X, X'], \xi) = \eta([X, X']) = -2d\eta(X, X') = 0,$$

hence $[X, X'] \in \Gamma(L)$, and in a similar manner one can prove the integrability of Q . Thus L and Q define two orthogonal Legendrian foliations on M and now the result follows from Theorem 4.1. \square

The connection $\tilde{\nabla}$ is, under certain points of view, an "invariant" of the contact metric (κ, μ) -space unless \mathcal{D} -homothetic deformations. Indeed, by a direct computation, one has the following result.

Proposition 4.2. *The bi-Legendrian connection associated to a contact metric (κ, μ) -space remains unchanged under a \mathcal{D} -homothetic deformation.*

The connection stated in Theorem 4.2 should be compared to the Tanaka-Webster connection of a non-degenerate integrable CR-manifold (cf. [17], [22]) and to the generalized Tanaka-Webster connection introduced by Tanno in [20]. This can be seen in the following theorem, where we prove, using Theorem 4.2, the already quoted result that any contact metric (κ, μ) -space is a strongly pseudo-convex CR-manifold.

Corollary 4.2. *Any contact metric (κ, μ) -space is a strongly pseudo-convex CR-manifold.*

Proof. We define a connection on M as follows. We put

$$\hat{\nabla}_V W = \begin{cases} \bar{\nabla}_V W, & \text{if } V \in \Gamma(\mathcal{D}); \\ -\phi hW + [\xi, W], & \text{if } V = \xi. \end{cases}$$

Then it easy to check that $\hat{\nabla}$ coincides with the Tanaka-Webster connection of M and so we get the assertion. \square

The above characterization may be also a tool for proving properties on (κ, μ) -spaces. As an application we show in a very simple way that an invariant submanifold of a contact metric (κ, μ) -space, that is a submanifold N such that $\phi T_p N \subset T_p N$ for all $p \in N$, is in turn a contact metric (κ, μ) -space (cf. [21]).

Corollary 4.3. *Any invariant submanifold of a contact metric (κ, μ) -space is in turn a (κ, μ) -space.*

Proof. It is well-known (cf. [3]) that an invariant submanifold of a contact metric manifold inherits a contact metric structure by restriction. Now let N^{2m+1} be an invariant submanifold of M^{2n+1} and consider the distribution on N given by $L_x := T_x N \cap \mathcal{D}_{\lambda x}$ and $Q_x := T_x N \cap \mathcal{D}_{-\lambda x}$ for all $x \in N$. It is easy to check that L and Q define two mutually orthogonal Legendrian foliations of N^{2m+1} and that the bi-Legendrian connection corresponding to (L, Q) is just the connection induced on N by the bi-Legendrian connection associated to $(\mathcal{D}_{\lambda}, \mathcal{D}_{-\lambda})$. The result now follows from Theorem 4.2. \square

We conclude showing that the assumption in Theorem 4.2 that ξ must be not Killing is essential. This can be seen in the following example.

Example 4.1. Consider the sphere $S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$ with the following Sasakian structure:

$$\eta = x_3 dx_1 + x_4 dx_2 - x_1 dx_3 - x_2 dx_4, \quad \xi = x_3 \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3} - x_2 \frac{\partial}{\partial x_4},$$

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \phi = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Set $X := x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} - x_4 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4}$ and $Y := \phi X = x_4 \frac{\partial}{\partial x_1} - x_3 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3} - x_1 \frac{\partial}{\partial x_4}$, and consider the 1-dimensional distributions L and Q on S^3 generated by X and Y , respectively. An easy computation shows that $[X, \xi] = -2Y$, $[Y, \xi] = 2X$, $[X, Y] = 2\xi$. Thus L and Q defines two non-degenerate, orthogonal Legendrian foliations on the Sasakian manifold $(S^3, \phi, \xi, \eta, g)$. For the bi-Legendrian connection corresponding to this bi-Legendrian structure, we have, after a straightforward computation, $\bar{\nabla}_X X = \bar{\nabla}_X Y = \bar{\nabla}_X \xi = 0$ and $\bar{\nabla}_Y X = \bar{\nabla}_Y Y = \bar{\nabla}_Y \xi = 0$. Therefore $\bar{\nabla} \phi = 0$ and so, by Proposition 2.2, also $\bar{\nabla} g = 0$. Moreover, as ξ is Killing obviously $\bar{\nabla} h = 0$.

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